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Minimum connected dominating sets and maximal independent sets in unit disk graphs

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Abstract

In ad hoc wireless networks, a connected dominating set can be used as a virtual backbone to improve the performance. Many constructions for approximating the minimum connected dominating set are based on the construction of a maximal independent set. The relation between the size $mis(G)$ of a maximum independent set and the size $cds(G)$ of a minimum connected dominating set in the same graph G plays an important role in establishing the performance ratio of those approximation algorithms. Previously, it is known that $mis(G) \leqslant 4 \cdot cds(G) + 1$ for all unit disk graphs G. In this paper, we improve it by showing $mis(G) \leqslant 3.8 \cdot cds(G) + 1.2$.

Keywords: Connected dominating set; Independent set; Unit disk graphs

1. Introduction

Wireless sensor networks have been widely used in many industries such as the healthcare industry, food industry, and agriculture [12]. Such a network consists of many sensors each of which is not only a mobile host but also a router. In other words, the sensors are able to forward the received data packages according to routing protocols. Usually, the sensors are cheap devices having an identical design. In this situation, every sensor has the same power and hence can communicate with others within a unit distance so that the topology of the sensor network can be formulated as a unit disk graph.

A unit disk is a disk with radius one. A *unit disk graph* is associated with a set of unit disks in the Euclidean plane. Each vertex is the center of a unit disk. An edge exists between two vertices u and v if and only if $|uv| \leqslant 1$ where $|uv|$ is the Euclidean distance between u and v. This means that two vertices are connected by an edge if and only if u's disk covers v and v's disk covers u.

A subset of vertices in a graph is called a *dominating set* if every vertex is either in the subset or adjacent to a vertex in the subset. A dominating set is *connected* if it induces a connected subgraph. A connected dominating set

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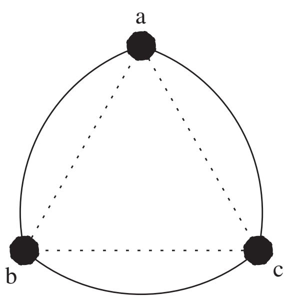

Fig. 1. Unit arc-triangle *abc*.

is often used as a virtual backbone in wireless sensor networks to improve communication and storage performance [8]. Clearly, the smaller virtual backbone gives the better performance. However, computing the minimum connected dominating set is NP-hard even in unit disk graphs. Therefore, many efforts [3,13–16,1,4,6] have been made to design approximations or heuristics for the minimum connected dominating set. Among distributed approximation algorithms, the best performance is achieved by algorithms [15,4,10] using a popular idea as follows: First, construct a dominating set and then connect it by adding more vertices.

Since every maximal independent set is a dominating set and it is easy to construct, one usually constructs a maximal independent set at the first step. Therefore, the approximation performance ratio would be determined by two facts. The first is how large a maximal independent set can be compared with a minimum connected dominating set. The second is how many vertices are required to connect a maximal independent set. It is shown in [15] that in every unit disk graph G,

$$mis(G) \leqslant 4 \cdot cds(G) + 1,$$

where $mis(G)$ is the size of a maximum independent set and $cds(G)$ is the size of a minimum connected dominating set in G. In this paper, we show that

$$mis(G) \leqslant 3.8 \cdot cds(G) + 1.2.$$

Therefore, all evaluation of performance ratios in [15,4,10] are improved.

2. Preliminary

We call a unit disk (including its boundary) at center x the *neighbor area* of x, denoted by $N(x)$. In general, for a graph G, its *neighbor area* $N(G)$ is defined to be the union of all neighbor areas of its vertices. Throughout this paper, the study is referred to a unit disk graph G. Therefore, two vertices u and v are said to be *adjacent* if $|uv| \leqslant 1$ and *independent* if $|uv| > 1$.

The following lemma can be found in [15].

Lemma 1. *The neighbor area of a vertex contains at most five independent vertices.*

This lemma is proved by noting the following fact: Suppose u and v are two independent vertices in the neighbor area $N(x)$ of a vertex x. Then we must have $\angle uxv > 60°$. In fact, if $\angle uxv \leqslant 60°$, then $|uv| \leqslant \max(|ux|, |vx|) \leqslant 1$, contradicting the independence of u and v. This elementary fact will be used later without mentioning.

For simplicity of speaking, we say that points $x_1, x_2, \ldots, x_k$ *counter-clockwisely* lying in $N(x)$, if $xx_1, xx_2, \ldots, xx_k$ lie in counter-clockwise direction around x. When $x_1, x_2, \ldots, x_k$ are independent, we have $\angle x_1xx_2 > 60°, \ldots, \angle x_{k-1}xx_k > 60°$ and $\angle x_kxx_1 > 60°$ and hence $k60° < 360°$. This implies $k < 6$, that is $k \leqslant 5$.

Now, consider three vertices a, b, c of a regular triangle with unit edge length. Connect two vertices a and b by an arc of unit radius at center c, connect two vertices b and c by an arc of unit radius at center a, and connect two vertices a and c by an arc of unit radius at center b. Let A be the area surrounded by the three arcs (Fig. 1). We call A a *unit* *arc-triangle abc*. It is a well-known fact that every two points in the area A have distance at most one. Therefore, we have

Lemma 2. *The unit arc-triangle A cannot contain two independent vertices.*

3. Main results

By Lemma 1, in the neighbor area of two adjacent vertices, there are at most nine independent vertices. However, the following lemma gives a better result.

Lemma 3. *The neighbor area of two adjacent vertices contains at most eight independent vertices.*

Proof. Consider two adjacent vertices u and v. For contradiction, suppose that the neighbor area of u and v contains an independent set I of more than eight vertices. First, we claim that the intersection $A = N(u) \cap N(v)$ contains exactly one vertex in I. In fact, if A contains k vertices in I, then by Lemma 1, $N(u) - A$ contains at most $5 - k$ vertices in I and $N(v) - A$ contains at most $5 - k$ vertices in I. Therefore, $N(u, v)$ contains at most $10 - k$ vertices in I. Hence, $10 - k \geqslant 9$, that is, $k \leqslant 1$. Let x and y be two intersection points of boundaries of $N(u)$ and $N(v)$. Since $|uv| \leqslant 1$, we have $\angle xvy = \angle yux \geqslant 120°$. Thus, $N(u) - A$ contains at most four vertices in I, so does $N(v) - A$. This means that I contains at most $8 + k$ vertices and hence $8 + k \geqslant 9$, that is, $k \geqslant 1$.

Let a_0 be the unique vertex in I, lying in $N(u) \cap N(v)$. As a consequence, each of $N(u) - A$ and $N(v) - A$ contains exactly four vertices in I and $|I| = 9$. Suppose $I = \{a_0, a_1, \ldots, a_8\}$, $a_0, a_1, \ldots, a_4$ lie counter-clockwisely in $N(u)$ and $a_0, a_5, \ldots, a_8$ lie counter-clockwisely in $N(v)$. Denote by ub_i the radius containing a_i for $i = 2, \ldots, 4$ and by vb_i the radius containing a_i for $i = 5, \ldots, 8$. Draw four unit arc-triangles ub_2c_2, ub_3c_3, vb_6c_6, and vb_7c_7 as shown in Fig. 2. Their boundaries intersect the boundary of $N(u) \cap N(v)$ at d_2, d_3, d_6, d_7, respectively. Note that none of a_1, a_4, a_5, a_8 can lie in the four unit arc-triangles and $N(u) \cap N(v)$. Therefore, a_1, a_4, a_5, a_8 must lie in the four small dark areas xc_2d_2, yc_3d_3, yc_6d_6 and xc_7d_7, respectively, as shown in Fig. 2.

Next, we will show that there exist two small dark areas too close to contain two independent vertices, a contradiction. To do so, we note that $\angle b_2ub_3 > 60°$ and $\angle c_2ub_2 = \angle b_3uc_3 = 60°$. Hence, $\angle c_2uc_3 > 180°$ and $\angle c_3uc_2 < 180°$ (here, please note that $\angle c_3uc_2$ is the one obtained by moving c_3u counterclockwisely to c_2u). Similarly, $\angle c_7vc_6 < 180°$. Therefore $\angle uc_2c_7 + \angle c_2c_7v + \angle vc_6c_3 + \angle c_6c_3u > 360°$. Hence, either $\angle uc_2c_7 + \angle c_2c_7v > 180°$ or $\angle vc_6c_3 + \angle c_6c_3u > 180°$. Assume the former occurs without loss of generality (Fig. 3). We show that dark areas xc_2d_2 and xc_7d_7 cannot contain two vertices in I.

To do so, we first enlarge area xc_7d_7 by turning the unit arc-triangle vb_7c_7 around v until vc_7 is parallel to uc_2. At this limit position, quadrilateral c_2uvc_7 becomes a parallelogram so that $|c_2c_7| = |uv| \leqslant 1$. It is easy to see that the distance between two points in areas xc_2d_2 and xc_7d_7 cannot exceed $\max(|c_2c_7|, |c_2d_7|, |d_2c_7|, |d_2d_7|)$. Moreover, we claim that $|d_2d_7| \leqslant \max(|c_2d_7|, |d_2c_7|)$. In fact, note that $\angle c_7d_7d_2 + \angle d_7d_2c_2 > 180°$. Thus, either $\angle c_7d_7d_2 > 90°$ or $\angle d_7d_2c_2 > 90°$. Therefore, either $|d_2c_7| > |d_2d_7|$ or $c_2d_7 > |d_2d_7|$, that is, our claim is true.

Now, to complete the proof of the lemma, it remains to prove that $|c_2d_7| \leqslant 1$ and $|d_2c_7| \leqslant 1$.

To see $|c_2d_7| \leqslant 1$, we first make $|c_2d_7|$ longer by moving v away from u until $|uv| = 1$ (Fig. 4). At this limit position, we have $|uv| = |vb_7| = |b_7d_7| = |d_7u| = 1$. Therefore, uvb_7d_7 is a parallelogram. Hence, $|d_7b_7| = |c_2c_7| = 1$ and d_7b_7 is parallel to uv and hence parallel to c_2c_7. It follows that $c_2d_7b_7c_7$ is a parallelogram. Thus, $|c_2d_7| = |c_7b_7| = 1$. Similarly, we can show $|d_2c_7| \leqslant 1$. □

The following two lemmas are about properties of graphs.

Lemma 4. *For any unit disk graph, there exists a minimum spanning tree such that every vertex has degree at most five.*

Proof. Let T be a minimum spanning tree. It is easy to see that T must have the following two properties:

(a1) Two edges meeting at a vertex form an angle of at least 60°.
(a2) If two edges form an angle of exactly 60°, then they have the same length.

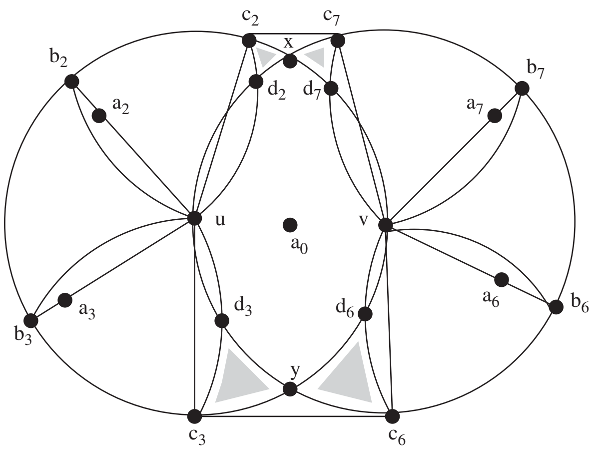

Fig. 2. Four small dark areas.

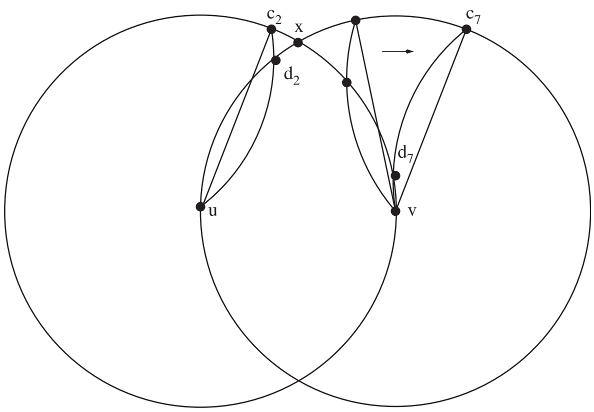

Fig. 3. Turn unit arc-triangle vb_7c_7 until $vc_7 \parallel uc_2$.

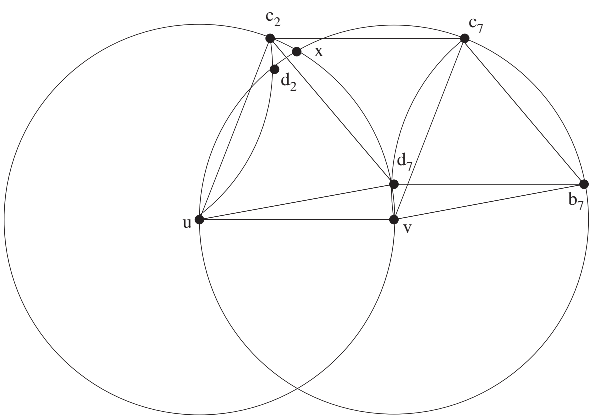

Fig. 4. Move u until $|uv| = 1$.

It follows immediately from (a1) that every vertex in T has degree at most six. Consider a vertex u with degree six in T. By (a1), every angle at u equals 60°. By (a2), all edges incident to u have the equal length. These two facts imply that every vertex v adjacent to a vertex u of degree six has degree at most four. In fact, u has two edges uw and ux such that $\angle wuv = \angle vux = 60°$ and $|uv| = |uw| = |ux|$. It follows that $|vw| = |uw|$ and $|vx| = |ux|$. Thus, replacing uw and ux by vw and vx results in still a minimum spanning tree. But, v gets two more edges in the new tree. Hence, v has degree at most four in the original tree. Now, if we do only one replacement, that is, replace uw by vw, but do not replace ux by vx. Then both u and v have degree at most five.

One may worry about that v may be adjacent to more than one vertices of degree six. In such a case, could v receives too many edges and its degree increases to six? This cannot happen because after all replacements, resulting tree is still a minimum spanning tree and hence it still has property that every vertex adjacent to a vertex of degree six has degree at most four. However, v would be adjacent to at least one vertex of degree five in the new tree. It follows that v cannot have degree six. □

Lemma 5. *Every tree T with at least three vertices has a non-leaf vertex adjacent to at most one non-leaf vertex.*

Proof. Let T' be the subtree obtained from T by removal of all leave. Since T has at least three vertices, T' contains at least one vertex. If T' contains only one vertex, then it meets our requirement. If T' contains more than one vertices, then every leaf of T' is a non-leaf vertex of T satisfying the condition stated in the lemma. □

Now, we are ready to show our main theorem.

Theorem 1. *For any unit disk graph G, the size of a maximal independent set is at most $3.8cds(G) + 1.2$ where $cds(G)$ is the size of a minimum connected dominating set.*

Proof. Let G be a subgraph induced by a minimum connected dominating set in the given unit disk graph. Then G is a unit disk subgraph. By Lemma 4, G has a minimum spanning tree T such that every vertex has degree at most five. Let $|T|$ denote the number of vertices in T. We will show by induction on $|T|$ that there exists at most $3.8|T| + 1.2$ independent vertices in the neighbor area of T. For $|T| = 1$ or 2, this is true by Lemmas 1 and 3. Next, we assume $|T| \geqslant 3$. By Lemma 5, T contains a non-leaf vertex v adjacent to at most one non-leaf vertex. Let u be the non-leaf neighbor of v if it exists, or a leaf neighbor of v, otherwise. Let $x_1, \ldots, x_k$ $(k \leqslant 4)$ be other neighbors of v. Note that for each x_i for $1 \leqslant i \leqslant k-1$ its neighbor area contains at most four independent vertices also independent from v by Lemma 1 and the neighbor area of v and x_k contains at most seven independent vertices also independent from u by Lemma 3. Moreover, by the induction hypothesis, the neighbor area of $T - \{v, x_1, \ldots, x_k\}$ contains at most $3.8(|T| - k - 1) + 1.2$ independent vertices. Therefore, the neighbor area of T contains at most

$$3.8(|T| - k - 1) + 1.2 + 7 + 4(k - 1) = 3.8|T| + 1.2 + 0.2(k - 4) \leqslant 3.8|T| + 1.2$$

independent vertices. Note that $|T| = cds(G)$. This completes the proof of the theorem. □

As a corollary, we have

Corollary 1. *For approximation algorithms in* [15,4] *for the minimum connected dominating set, the performance ratio can be reduced from* 8 *to* 7.8.

4. Discussion

A 4-star is a graph with a center and four leaves. We believe the conjecture that

Conjecture 1. *The neighbor area of a* 4*-star subgraph in a unit disk graph contains at most* 20 *independent vertices.*

If this true, then by a similar argument, Theorem 1 can be improved from 3.8 to 3.6. However, dealing with 20 points with elementary geometric method is quite hard. Therefore, the proof of this conjecture may need some advanced

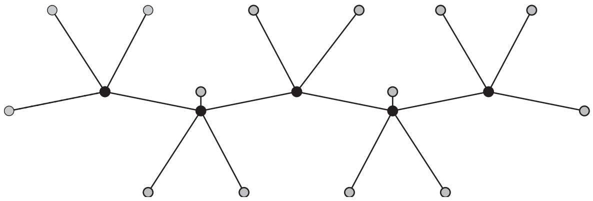

Fig. 5. A class of graph for $mis(G) = 2.5 \cdot cds(G) + 1.5$.

methods for packing and covering, such as harmonic analysis. In fact, this conjecture can be easily transformed to a unit disk packing problem if we double the radius for those disks in construction of the neighbor area.

A *weakly connected dominating set* is a dominating set such that putting edges between dominers and edges between dominers and dominees results in a connected graph [5,2]. The weakly connected dominating set has also been used in wireless networks. In [2], it is showed that some special constructed maximal independent sets can be weakly connected. By Lemma 1, those maximal independent sets are approximation solutions within a factor of 5 from the minimum weakly connected dominating set. Could we improve this factor with Lemma 3? It is hard to answer since it is possible, but, not easy to obtain such an improvement. The difficulty is that two dominers may be connected through a dominee which is not in considered maximal independent set.

What is the smallest α such that $mis(G) \leqslant \alpha \cdot cds(G) + \beta$ for all unit disk graphs G and a constant β? Fig. 5 shows an infinite class of unit disk graphs G satisfying $mis(G) = 2.5 \cdot cds(G) + 1.5$. Therefore, $2.5 \leqslant \alpha \leqslant 3.8$. This gap leaves an open problem for further study.

Could a similar relation between the minimum connected dominating set and the maximal independent set hold in disk graphs? This is an important open problem. In fact, if such a relation is established, then it can imply the existence of constant-bounded polynomial-time approximations for the minimum connected dominating set in disk graph. Actually, it is a long-standing open problem whether there exists or not a constant-bounded polynomial-time approximation for the minimum connected dominating set in disk graph.

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